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# Singularly Perturbed Boundary Value Problem for Linear Equations with Turning Points

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In this paper, the boundary value problems for a class of linear ordinary differential equations with turning points are studied. Under suitable assumptions, the author proves the existence and uniqueness of solutions, and obtains the uniformly valid asymptotic expansions of the solutions. © 1991 Academic Press, Inc.

## 1. INTRODUCTION

For the boundary value problem with a small parameter $\varepsilon > 0$

$$\varepsilon y'' + f(x, \varepsilon) y' + g(x, \varepsilon) y = 0,$$

$$y(-a) = A(\varepsilon), \qquad y(b) = B(\varepsilon),$$

where $a$, $b > 0$ and $f(0, 0) = 0$, that is, $x = 0$ is a turning point, under the assumption that $f'(0, 0) < 0$, the asymptotic expansions of solutions have been studied in [1–4], respectively. For the case $f'(0, 0) > 0$ and $g(0, 0) = 0$, the uniformly valid asymptotic expansions of the solutions have been obtained in [5]. In this paper, under the assumption that $f(x, 0) < 0$ in $[-a, 0)$; $f(x, 0) > 0$ in $(0, b]$; $g(0, 0) < 0$, for more general linear equations the author studies the existence and uniquenness of the solutions, and obtains the uniformly valid asymptotic expansions of the solutions.

We are able to find that in general cases the solution of the boundary value problem possesses interior layers at the turning point and has expansions with powers of $\varepsilon^{1/2}$. Thus, consider the more general boundary value problem

$$\varepsilon^2 y'' + f(x, \varepsilon) y' + g(x, \varepsilon) y + h(x, \varepsilon) = 0, \tag{1}$$

$$y(-a) = A(\varepsilon), \qquad y(b) = B(\varepsilon), \tag{2}$$



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where $f(0, 0) = 0$, that is, $x = 0$ is the turning point. We make the following assumptions.

(I) $f(x, \varepsilon)$, $g(x, \varepsilon)$ and $h(x, \varepsilon) \in C^{(n+1)}([-a, b] \times [0, \varepsilon_0])$ ($\varepsilon_0$ is some positive constant), and $A(\varepsilon)$, $B(\varepsilon) \in C^{(n+1)}([0, \varepsilon_0])$. Thus, we have

$$f(x, \varepsilon) = \sum_{i=0}^{n} f_i(x)\, \varepsilon^i + O(\varepsilon^{n+1}),$$

$$g(x, \varepsilon) = \sum_{i=0}^{n} g_i(x)\, \varepsilon^i + O(\varepsilon^{n+1}),$$

$$h(x, \varepsilon) = \sum_{i=0}^{n} h_i(x)\, \varepsilon^i + O(\varepsilon^{n+1}),$$

$$A(\varepsilon) = \sum_{i=0}^{n} A_i \varepsilon^i + O(\varepsilon^{n+1}),$$

$$B(\varepsilon) = \sum_{i=0}^{n} B_i \varepsilon^i + O(\varepsilon^{n+1}),$$

where $f_i(x) = (1/i!)[(\partial^i/\partial\varepsilon^i)\, f(x, \varepsilon)]|_{\varepsilon=0}$, the functions $g_i(x)$ and the constants $A_i$, $B_i$ are similar to $f_i(x)$, respectively, and $n$ is an arbitrarily given nonnegative integer.

(II) $f_0(x) < 0$ on $[-a, 0)$; $f_0(x) > 0$ on $(0, b]$, and hence, $f_0'(0) \geq 0$. Moreover, $g_0(0) < 0$.

## 2. Constructing Formal Solution

Before constructing the formal solution, we prove the following theorem.

Theorem 1. *Under assumptions* (I) *and* (II), *the equations*

$$f_0(x)\, y_i + g_0(x)\, y_i + C_i(x) = 0 \qquad (i = 0, 1, 2, ..., n) \tag{3}$$

*have solutions* $\bar{y}_i(x)$ *on* $[0, b]$ *with* $\bar{y}_i(b) = B_i$ *and* $\bar{\bar{y}}_i(x)$ *on* $[-a, 0]$ *with* $\bar{\bar{y}}_i(-a) = A_i$ *satisfying*

$$\bar{y}_i(0) = -C_i(0+0)/g_0(0), \qquad \bar{\bar{y}}_i(0) = -C_i(0-0)/g_0(0),$$

*where* $C_i(x) = h_i(x) + y''_{i-2}(x) + \sum_{j=1}^{i} [f_j(x)\, y'_{i-j}(x) + g_j(x)\, y_{i-j}(x)]$ *in which the functions with negative subscripts are considered as zero.*

*Proof.* It is enough to prove that the equation

$$f_0(x)\, y' + g_0(x)\, y + H(x) = 0$$

has solutions $y_1(x)$ on $[0, b]$ with $y_1(b) = a_1$ and $y_2(x)$ on $[-a, 0]$ with $y_2(-a) = a_2$ satisfying

$$y_1(0) = -H(0+0)/g_0(0), \qquad y_2(0) = -H(0-0)/g_0(0),$$

where $H(x)$ is a continuous solution on $[-a, 0)$ or $(0, b]$ satisfying that $H(0+0)$ and $H(0-0)$ are finite, and $a_1$, $a_2$ are given constants.

It is clear that $y_1(x)$ on $(0, b]$ and $y_2(x)$ on $[-a, 0)$ as follows

$$y_i(x) = \exp\left[-\int_{b_i}^{x} \frac{g_0(x)}{f_0(x)}\, dx\right] \times \left(a_i - \int_{b_i}^{x} \frac{H(x)}{f_0(x)} \exp\left[\int_{b_i}^{x} \frac{g_0(x)}{f_0(x)}\, dx\right] dx\right),$$

where $b_1 = b$, $b_2 = -a$. Since $f_0(x) = f_0'(0)x + o(x)$ $(x \to 0)$, the integral $\int_{b_i}^{0} (g_0(x)/f_0(x))\, dx$ is divergent. So,

$$\begin{aligned}\lim_{x \to 0^+} y_1(x) &= -\lim_{x \to 0^+} \frac{\int_b^x (H(x)/f_0(x)) \exp[\int_b^x (g_0(x)/f_0(x))\, dx]\, dx}{\exp[\int_b^x (g_0(x)/f_0(x))\, dx]} \\ &= -\lim_{x \to 0^+} \frac{(H(x)/f_0(x)) \exp[\int_b^x (g_0(x)/f_0(x))\, dx]}{(g_0(x)/f_0(x)) \exp[\int_b^x (g_0(x)/f_0(x))\, dx]} \\ &= -H(0+0)/g_0(0).\end{aligned}$$

Similarly,

$$\lim_{x \to 0^-} y_2(x) = -H(0-0)/g_0(0).$$

Therefore, $y_1(x)$ on $[0, b]$ and $y_2(x)$ on $[-a, 0]$ satisfy

$$y_1(0) = -H(0+0)/g_0(0), \qquad y_2(0) = -H(0-0)/g_0(0).$$

The proof is complete.

Theorem 1 shows that under assumptions (I) and (II), solutions of the problem (1), (2) do not possess shock layers at the origin. Therefore we assume that the solution of the problem (1), (2) has the following formal expansion

$$y(x, \varepsilon) = \begin{cases} \bar{y}(x, \varepsilon) + \xi(\tau, \varepsilon), & x \in [0, b], \\ \bar{\bar{y}}(x, \varepsilon) + \eta(\tau, \varepsilon), & x \in [-a, 0], \end{cases} \tag{4}$$

where $\tau = x/\varepsilon$ and

$$\begin{aligned}
\bar{y}(x, \varepsilon) &= \bar{y}_0(x) + \bar{y}_1(x)\varepsilon + \bar{y}_2(x)\varepsilon^2 + \cdots, \\
\xi(\tau, \varepsilon) &= \xi_1(\tau)\varepsilon + \xi_1(\tau)\varepsilon^2 + \xi_3(\tau)\varepsilon^3 + \cdots, \\
\bar{\bar{y}}(x, \varepsilon) &= \bar{\bar{y}}_0(x) + \bar{\bar{y}}_1(x)\varepsilon + \bar{\bar{y}}_2(x)\varepsilon^2 + \cdots, \\
\eta(\tau, \varepsilon) &= \eta_1(\tau)\varepsilon + \eta_2(\tau)\varepsilon^2 + \eta_3(\tau)\varepsilon^2 + \cdots,
\end{aligned} \tag{5}$$

in which the functions $\xi_i(\tau)$, $\eta_i(\tau)$, $\bar{y}_i(x)$, and $\bar{\bar{y}}_i(x)$ remain to be determined.

Substituting (4) into Eq. (1), we have, on $[0, b]$,

$$\begin{aligned}
&\varepsilon^2 \bar{y}''(x, \varepsilon) + f(x, \varepsilon)\, \bar{y}'(x, \varepsilon) + g(x, \varepsilon)\, \bar{y}(x, \varepsilon) + h(x, \varepsilon) \\
&\quad + \xi''(\tau, \varepsilon) + \frac{1}{\varepsilon} f(\tau\varepsilon, \varepsilon)\, \xi'(\tau, \varepsilon) + g(\tau\varepsilon, \varepsilon)\, \xi(\tau, \varepsilon) = 0,
\end{aligned}$$

and on $[-a, 0]$,

$$\begin{aligned}
&\varepsilon^2 \bar{\bar{y}}''(x, \varepsilon) + f(x, \varepsilon)\, \bar{\bar{y}}'(x, \varepsilon) + g(x, \varepsilon)\, \bar{\bar{y}}(x, \varepsilon) + h(x, \varepsilon) \\
&\quad + \eta''(\tau, \varepsilon) + \frac{1}{\varepsilon} f(\tau\varepsilon, \varepsilon)\, \eta'(\tau, \varepsilon) + g(\tau\varepsilon, \varepsilon)\, \eta(\tau, \varepsilon) = 0.
\end{aligned}$$

Since $\tau$ is a stretched variable, we consider $x$ and $\tau$ as independent, and set

$$\begin{aligned}
\varepsilon^2 \bar{y}''(x, \varepsilon) + f(x, \varepsilon)\, \bar{y}'(x, \varepsilon) + g(x, \varepsilon)\, \bar{y}(x, \varepsilon) + h(x, \varepsilon) &= 0 \qquad (0 \leqslant x \leqslant b), \\
\xi''(\tau, \varepsilon) + \frac{1}{\varepsilon} f(\tau\varepsilon, \varepsilon)\, \xi'(\tau, \varepsilon) + g(\tau\varepsilon, \varepsilon)\, \xi(\tau, \varepsilon) &= 0 \qquad (0 \leqslant \tau < \infty), \\
\varepsilon^2 \bar{\bar{y}}''(x, \varepsilon) + f(x, \varepsilon)\, \bar{\bar{y}}'(x, \varepsilon) + g(x, \varepsilon)\, \bar{\bar{y}}(x, \varepsilon) + h(x, \varepsilon) &= 0 \qquad (-a \leqslant x \leqslant 0), \\
\eta''(\tau, \varepsilon) + \frac{1}{\varepsilon} f(\tau\varepsilon, \varepsilon)\, \eta'(\tau, \varepsilon) + g(\tau\varepsilon, \varepsilon)\, \eta(\tau, \varepsilon) &= 0 \qquad (-\infty < \tau \leqslant 0).
\end{aligned}$$

Substituting (5) into the above equations, collecting the terms of like powers of $\varepsilon$ and equating the coefficients to zero, we have

$$f_0(x)\, \bar{y}_i'(x) + g_0(x)\, \bar{y}_i(x) + \bar{C}_i(x) = 0 \qquad (0 \leqslant x \leqslant b), \tag{6$_i$}$$

$$\xi_i''(\tau) + [\tau f_0'(0) + f_1(0)]\, \xi_i'(\tau) + g_0(0)\, \xi_i(\tau) = R_i(\tau) \qquad (0 \leqslant \tau < \infty), \tag{7$_i$}$$

$$f_0(x)\,\bar{\bar{y}}_i''(x)+g_0(x)\,\bar{\bar{y}}_i(x)+\bar{\bar{C}}_i(x)=0$$
$$(-a\leqslant x\leqslant 0), \qquad (8)_i$$

$$\eta_i''(\tau)+[\tau f_0'(0)+f_1(0)]\,\eta_i'(\tau)+g_0(0)\,\eta_i(\tau)=H_i(\tau)$$
$$(-\infty<\tau\leqslant 0), \qquad (9)_i$$

where $\bar{C}_i(x)$ and $\bar{\bar{C}}_i(x)$ are the $C_i(x)$ in (3) in which $y_k$ $(k=0, 1, 2, ..., i-1)$ is replaced by $\bar{y}_k(x)$ and $\bar{\bar{y}}_k(x)$, respectively, $\xi_0$, $\eta_0$, $R_0$, and $H_0$ are considered as zero, and $R_1=H_1=0$, for $i=2, 3, ..., n$,

$$R_i(\tau)=-\sum_{j=1}^{i}\,[F_{j+1}(\tau)\,\xi_{i-j}'(\tau)+G_j(\tau)\,\xi_{i-j}(\tau)],$$

$$H_i(\tau)=-\sum_{j=1}^{i}\,[F_{j+1}(\tau)\,\eta_{i-j}'(\tau)+G_j(\tau)\,\eta_{i-j}(\tau)],$$

in which

$$F_i(\tau)=\frac{1}{i!}\left[\frac{d^i}{d\varepsilon^i}f(\tau\varepsilon,\varepsilon)\right]\Bigg|_{\varepsilon=0}, \qquad G_i(\tau)=\frac{1}{i!}\left[\frac{d^i}{d\varepsilon^i}g(\tau\varepsilon,\varepsilon)\right]\Bigg|_{\varepsilon=0}.$$

In order to obtain $\bar{y}_i(x)$, $\bar{\bar{y}}_i(x)$, $\xi_i(\tau)$, and $\eta_i(\tau)$ $(i=0, 1, ..., n)$ from $(6)_i$–$(9)_i$, we need some suitable boundary conditions. The fact that $\xi(\tau,\varepsilon)$ and $\eta(\tau,\varepsilon)$ are transition layer functions yields

$$\xi_i(\infty)=0, \qquad \eta_i(-\infty)=0. \qquad (10)_i$$

In addition, the formal solution (4) must satisfy the condition (2) and some initial value conditions at $x=0$. Thus, it follows that

$$\bar{y}_i(b)=B_i, \qquad \bar{\bar{y}}_i(-a)=A_i, \qquad (11)_i$$

$$\begin{aligned}\bar{y}_i(0)+\xi_i(0)&=\bar{\bar{y}}_i(0)+\eta_i(0),\\ \bar{y}_{i-1}'(0)+\xi_i'(0)&=\bar{\bar{y}}_{i-1}'(0)+\eta_i'(0),\end{aligned} \qquad (12)_i$$

where $i=0, 1, 2, ..., n$, and the terms with negative subscripts are considered as zero. Now we prove a result as follows.

THEOREM 2. *Under assumptions* (I) *and* (II), *there exists a unique solution* $\bar{y}_i(x)$ *on* $[-a, 0]$, $\bar{\bar{y}}_i(x)$ *on* $[0\ b]$ *and* $\xi_i(\tau)$ *on* $[0\ \infty)$, $\eta_i(\tau)$ *on* $(-\infty, 0]$ *satisfying* $(6)_i$–$(12)_i$, *respectively. In addition, there exists a positive constant* $\sigma$ *such that*

$$\xi_i(\tau)=O(e^{-\sigma\tau}) \quad (\tau\to\infty), \qquad \eta_i(\tau)=O(e^{\sigma\tau}) \quad (\tau\to-\infty). \qquad (13)_i$$

Before proving Theorem 2, we give a lemma as follows.

LEMMA 1. *Assume that the infinite boundary value problem*

$$y'' = f(x)\,y + g(x), \qquad -\infty < x < \infty, \tag{14}$$

$$y(0) = y_0, \qquad y(\infty) = 0 \qquad (or\ y(-\infty) = 0), \tag{15}$$

*satisfies*

(i) *there exists a constant* $m > 0$ *such that* $f(x) > m^2$ *on* $(-\infty, \infty)$;

(ii) *the functions* $f(x)$ *and* $g(x)$ *are continuous on* $(-\infty, \infty)$ *and* $g(x) = O(e^{-mx})$ $(x \to \infty)$ *(or* $g(x) = O(e^{mx})$ $(x \to -\infty)$*). Then, the infinite boundary value problem* (14), (15) *has a unique solution* $y(x) = O(e^{-m(1-\delta)x})$ $(x \to \infty)$ *(or* $y(x) = O(e^{m(1-\delta)x})$ $(x \to -\infty)$*), where* $\delta$ *is an arbitrary positive constant.*

*Proof.* We shall first prove that the fundamental system of the solutions of the equation

$$y'' = f(x)\,y, \tag{16}$$

cosists of a solution decaying exponentially as $x \to \infty$ and growing exponentially as $x \to -\infty$ and a solution growing exponentially as $x \to \infty$ and decaying exponentially as $x \to -\infty$. To do this, we consider the equation

$$y'' = m^2 y,$$

which has two linearly independent solutions $e^{-mx}$ and $e^{mx}$. It is clear that $\varphi_1(x) \triangleq 0 < \varphi_2(x) \triangleq e^{-mx}$, and

$$\varphi_1'' = m^2 \varphi_1 = f(x)\,\varphi_1,$$

$$\varphi_2'' = m^2 \varphi_2 < f(x)\,\varphi_2.$$

By [7, Theorem 1.7.1] Eq. (16) has a solution $y_1(x)$ on $[0, \infty)$ satisfying $y_1(0) = 1$ and $\varphi_1(x) \leqslant y_1(x) \leqslant \varphi_2(x)$ on $[0, \infty)$. In fact, $0 < y_1(x) \leqslant \varphi_2(x)$ on $[0, \infty)$ holds. Otherwise, it contradicts the uniqueness of solutions of initial value problems for Eq. (16). Thus, let $y_1(x) = e^{-p(x)}$, where $p(x) = -\ln y_1(x)$ has continuous derivatives up to second order.

Similarly, the equation

$$\bar{y}'' = f(-x)\,\bar{y}$$

has a solution $\bar{y}(x)$ on $[0, \infty)$ satisfying $\bar{y}(0) = 1$ and $0 < \bar{y}(x) \leqslant e^{-mx}$ on $[0, \infty)$, therefore, the function $y_2(x) \triangleq \bar{y}(-x)$ on $(-\infty, 0]$ is a solution of Eq. (16) such that $y_2(0) = 1$ and $0 < y_2(x) \leqslant e^{mx}$ on $(-\infty, 0]$. Let $y_2(x) = e^{q(x)}$, where $q(x) = \ln y_2(x)$ has continuous derivatives up to second order.

It is clear that the solutions $y_1(x)$ and $y_2(x)$ can be extended to $(-\infty, \infty)$, respectively, satisfying $y_1(x) \geqslant e^{-mx}$ on $(-\infty, 0]$ and $y_2(x) \geqslant e^{mx}$ on $[0, \infty)$. Now let $y_1(x) = e^{-p(x)}$ on $(-\infty, 0]$ and $y_2(x) = e^{q(x)}$ on $[0, \infty)$, where $p(x) = -\ln y_1(x)$ and $q(x) = \ln y_2(x)$ have continuous derivatives up to second order. Now we obtain the two linearly independent solutions $y_1(x)$ and $y_2(x)$ of (16) with behaviors the same as those stated above.

Next, we shall prove that $y_1(x) \cdot y_2(x) = e^{q(x) - p(x)}$ is bounded on $(-\infty, \infty)$. The proof is given only for the case for $[0, \infty)$ because the case for $(-\infty, 0]$ can be proved similarly. It follows from Liouville's Theorem that

$$\begin{vmatrix} e^{-p(x)} & e^{q(x)} \\ -p'(x)\, e^{-p(x)} & q'(x)\, e^{q(x)} \end{vmatrix} = K \text{ (a constant)} \neq 0.$$

that is,

$$(p'(x) + q'(x))\, e^{q(x) - p(x)} = K.$$

In addition, we find that $p'(x) > 0$, $q'(x)$ satisfies the equation $u' + u^2 = f(x)$ and $q'(0) \geqslant m$. This implies $q'(x) \geqslant m$ on $[0, \infty)$. If this is not true, there exists some point $x_0 \in (0, \infty)$ such that $q'(x_0) < m$. Let $x = x_1$ be the maximal zero point of the function $q'(x) - m$ at the left of the point $x_0$. Then there exists at least one point $x_2 \in (x_1, x_0)$ such that $q''(x_2) < 0$, but this contradicts $q''(x_2) = f(x_2) - q'^2(x_2) > 0$. Thus, $p'(x) + q'(x) > m$ holds on $[0, \infty)$, therefore $e^{q(x) - p(x)}$ is bounded on $[0, \infty)$.

Thus we obtain the general solutions of (14) as follows

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) + \int_0^x (y_1(t)\, y_2(x) - y_1(x)\, y_2(t))\, K^{-1} g(t)\, dt \\ &= c_1 y_1(x) + c_2 y_2(x) + K^{-1} y_2(x) \int_0^x y_1(t)\, g(t)\, dt \\ &\quad - K^{-1} y_1(x) \int_0^x y_2(t)\, g(t)\, dt \\ &= c_1 y_1(x) + c_2 y_2(x) \\ &\quad + K^{-1} y_2(x) \left( \int_0^\infty y_1(t)\, g(t)\, dt + \int_\infty^x y_1(t)\, g(t)\, dt \right) \\ &\quad - K^{-1} y_1(x) \cdot O(y_2(x) \cdot \exp[-m(1-\delta)x]) \qquad (x \to \infty) \end{aligned}$$

$$\begin{aligned}
&= c_1 y_1(x) + c_2 y_2(x) \\
&\quad + K^{-1} y_2(x)(L + O(y_1(x)\cdot \exp[-m(1-\delta)x])(x \to \infty)) \\
&\quad + O(\exp[-m(1-\delta)x]) \qquad (x \to \infty) \\
&= c_1 y_1(x) + (c_2 + K^{-1}L)\, y_2(x) + O(\exp[-m(1-\delta)x]) \qquad (x \to \infty).
\end{aligned}$$

It follows that $y(\infty)=0$ holds if and only if $c_2 = -K^{-1}L$. By $y(0)=y_0$, $c_1$ can be uniquely determined. Thus, the problem (14), (15) has a unique solution $y(x) = O(\exp[-m(1-\delta)x])$ $(x \to \infty)$.

Similarly, for the other infinite boundary value problem, we can obtain the solutions with $y(-\infty)=0$ as follows:

$$y(x) = c_2 y_2(x) + O(\exp[m(1-\delta)x]) \qquad (x \to -\infty).$$

By $y(0)=y_0$, $c_2$ can be uniquely determined. Thus there exists a unique solution $y(x) = O(\exp[m(1-\delta)x])$ $(x \to -\infty)$. The proof of Lemma 1 is complete.

*Proof of Theorem* 2. By Theorem 1, we can uniquely obtain $\bar{y}_i(x)$ on $[0, b]$ and $\bar{\bar{y}}_i(x)$ on $[-a, 0]$ successively. In order to obtain $\xi_i(\tau)$ and $\eta_i(\tau)$, we introduce new variables $u_i(\tau)$ and $\bar{u}_i(\tau)$ such that $\xi_i(\tau) = v(\tau)\cdot u_i(\tau)$, $\eta_i(\tau) = v(\tau)\cdot \bar{u}_i(\tau)$, where

$$v(\tau) = \exp[-\tfrac{1}{4}(f_0'(0)\tau^2 + 2f_1(0)\tau)].$$

Then, Eqs. $(7)_i$, $(9)_i$ are respectively transformed into

$$u_i'' + [g_0(0) - \tfrac{1}{2}f_0'(0) - \tfrac{1}{4}(\tau f_0'(0) + f_1(0))^2]\,u_i = v^{-1}(\tau)\, R_i(\tau), \qquad (17)_i$$

$$\bar{u}_i'' + [g_0(0) - \tfrac{1}{2}f_0'(0) - \tfrac{1}{4}(\tau f_0'(0) + f_1(0))^2]\,\bar{u}_i = v^{-1}(\tau)\, H_i(\tau). \qquad (18)_i$$

Now we shall discuss the solutions of $(17)_i$, $(18)_i$ on $(-\infty, \infty)$. By Lemma 1, Eqs. $(17)_1$ and $(18)_1$ have solutions $u(\tau)$ and $\bar{u}(\tau)$, the behaviors of which are the same as those of $y_1(x)$ and $y_2(x)$ in Lemma 1. Thus, the solutions of $(17)_1$ with $u_1(\infty)=0$ and the solutions of $(18)_1$ with $\bar{u}_1(-\infty)=0$ are respectively

$$u_1(\tau) = c_1 u(\tau), \qquad \bar{u}_1(\tau) = \bar{c}_1 \bar{u}(\tau),$$

where $c_1$ and $\bar{c}_1$ are constants to be determined. Let $\tilde{u}(\tau) = v\cdot u(\tau)$ and $\tilde{\bar{u}}(\tau) = v\cdot \bar{u}(\tau)$, then

$$\xi_1(\tau) = c_1 \tilde{u}(\tau), \qquad \eta_1(\tau) = \bar{c}_1 \tilde{\bar{u}}(\tau).$$

From $(12)_1$, we have

$$\begin{aligned} c_1\tilde{u}(0)-\bar{c}_1\tilde{\bar{u}}(0)&=\bar{\bar{y}}_1(0)-\bar{y}_1(0)\\ c_1\tilde{u}'(0)-\bar{c}_1\tilde{\bar{u}}'(0)&=\bar{\bar{y}}_0'(0)-\bar{y}_0'(0). \end{aligned} \tag{19}$$

Since $\tilde{u}(\tau)$ and $\tilde{\bar{u}}(\tau)$ are two linearly independent solutions of $(7)_1$, the determinant of the coefficients of the system (19) does not vanish. Thus $c_1$ and $\bar{c}_1$ can be uniquely obtained from (19), so we obtain

$$\xi_1(\tau)=v(\tau)\cdot O(\exp[-(-g_0(0)+\tfrac{1}{4}f_1^2(0))^{1/2}(1-\delta_1)\tau]) \qquad (\tau\to\infty),$$

$$\eta_1(\tau)=v(\tau)\cdot O(\exp[(-g_0(0)+\tfrac{1}{4}f_1^2(0))^{1/2}(1-\delta_1)\tau]) \qquad (\tau\to-\infty),$$

henceforth $\delta_i$ $(i=1,2,...)$ are arbitrary positive constants with $\delta_j<\delta_{j+1}$. It is clear that the estimates $(13)_1$ hold.

With the defined $\xi_1(\tau)$ and $\eta_1(\tau)$ the terms at right sides of $(17)_2$ and $(18)_2$ are $O(\exp[-(-g_0(0)+\frac{1}{4}f_1^2(0))^{1/2}(1-\delta_2)\tau])$ $(\tau\to\infty)$ and $O(\exp[(-g_0(0)+\frac{1}{4}f_1^2(0))^{1/2}(1-\delta_2)\tau])$ $(\tau\to-\infty)$, respectively. It follows from Lemma 1 that the solutions of $(17)_2$ with $u_2(\infty)=0$ and the solutions of $(18)_2$ with $\bar{u}_2(-\infty)=0$ are, respectively,

$$u_2(\tau)=c_2u(\tau)+O(\exp[-(-g_0(0)+\tfrac{1}{4}f_1^2(0))^{1/2}(1-\delta_3)\tau]) \qquad (\tau\to\infty)$$

and

$$\bar{u}_2(\tau)=\bar{c}_2\bar{u}(\tau)+O(\exp[(-g_0(0)+\tfrac{1}{4}f_1^2(0))^{1/2}(1-\delta_3)\tau]) \qquad (\tau\to-\infty),$$

where $c_2$ ands $\bar{c}_2$ are constants to be determined, thus

$$\begin{aligned} \xi_2(\tau)&=c_2\tilde{u}(\tau)+v(\tau)\\ &\quad\cdot O(\exp[-(-g_0(0)+\tfrac{1}{4}f_1^2(0))^{1/2}(1-\delta_3)\tau]) \qquad (\tau\to\infty),\\ \eta_2(\tau)&=\bar{c}_2\tilde{\bar{u}}(\tau)+v(\tau)\\ &\quad\cdot O(\exp[(-g_0(0)+\tfrac{1}{4}f_1^2(0))^{1/2}(1-\delta_3)\tau]) \qquad (\tau\to-\infty). \end{aligned}$$

From $(12)_2$ we have

$$\begin{aligned} c_2\tilde{u}(0)-\bar{c}_2\tilde{\bar{u}}(0)&=\tilde{R},\\ c_2\tilde{u}'(0)-\bar{c}_2\tilde{\bar{u}}'(0)&=\tilde{\tilde{R}}, \end{aligned}$$

where $\tilde{R}$ and $\tilde{\tilde{R}}$ represent definite constants. Since the determinant of the coefficients of the above system does not vanish, $c_2$ and $\bar{c}_2$ can be uniquely obtained from the above system. Thus we obtain $\xi_2(\tau)$ and $\eta_2(\tau)$. It is clear that the estimates $(13)_2$ hold.

Similarly, $\xi_i(\tau)$ and $\eta_i(\tau)$ can be obtained from $(7)_i$, $(9)_i$, $(10)_i$, and $(12)_i$ successively such that the estimates $(13)_i$ hold. The proof of Theorem 2 is now completed.

## 3. Estimate of the Remainders

In order to prove Theorem 3 in this section, we need a lemma as follows

LEMMA 2. *Let $\alpha(x)$ and $\beta(x)$ be continuous functions on $[-a, b]$ satisfying*

(i) $\alpha(x) < \beta(x)$ *on* $[-a, b]$;

(ii) *there exists points $x_0, x_1, ..., x_n$ with $-a = x_0 < x_1 < \cdots < x_n = b$ such that $\alpha(x)$ and $\beta(x)$ have continuous derivatives up to second order on each interval $[x_{i-1}, x_i]$ $(i = 1, 2, ..., n)$, and for $i = 1, 2, ..., n$ $(n > 1)$,*

$$\alpha'(x_i - 0) \leqslant \alpha'(x_i + 0), \qquad \beta'(x_i - 0) \geqslant \beta'(x_i + 0)$$

*hold.*

*If $f(x)$, $g(x)$, and $h(x)$ are continuous in $[-a, b]$, and*

$$\alpha''(x) + f(x)\,\alpha'(x) + g(x)\,\alpha(x) + h(x) > 0,$$

$$\beta''(x) + f(x)\,\beta'(x) + g(x)\,\beta(x) + h(x) < 0,$$

*hold in $[x_{i-1}, x_i]$ $(i = 1, 2, ..., n)$, then for arbitrary real numbers $A$ and $B$, if only*

$$\alpha(-a) < A < \beta(-a), \qquad \alpha(b) < B < \beta(b),$$

*the boundary value problem*

$$y'' + f(x)\,y' + g(x)\,y + h(x) = 0, \qquad -a < x < b,$$
$$y(-a) = A, \qquad y(b) = B,$$

*has a unique solution $y(x)$ satisfying*

$$\alpha(x) < y(x) < \beta(x), \qquad -a \leqslant x \leqslant b. \tag{20}$$

*Proof.* The existence of solutions satisfying (20) can be directly obtained from [6, Theorem 2].

Now we prove the uniqueness. Let $y(x)$ be a solution satisfying (20). Suppose that the boundary value problem has another solution $y_0(x)$.

Then

$$y(x, c) = y(x) + c(y(x) - y_0(x)),$$

where $c$ is an arbitrary real number, are solutions of the boundary value problem. Consider the set $\{c; \alpha(x) < y(x, c) < \beta(x) \text{ in } [-a, b]\}$. It is clear that there exists a real number $c^*$ such that $\alpha(x) \leqslant y(x, c^*) \leqslant \beta(x)$ in $[-a, b]$ holds and there exists $x^*$ in $(-a, b)$ such that $y(x^*, c^*) = \alpha(x^*)$ or $y(x^*, c^*) = \beta(x^*)$. For definition, we consider the case that $y(x^*, c^*) = \alpha(x^*)$. It is easy to see that

$$y'(x^*, c^*) \leqslant \alpha'(x^* - 0) \leqslant \alpha'(x^* + 0) \leqslant y'(x^*, c^*)$$

thus, $y'(x^*, c^*) = \alpha'(x^*)$. Then we have

$$\begin{aligned} 0 &= y''(x^*, c^*) + f(x^*)\, y'(x^*, c^*) + g(x^*)\, y(x^*, c^*) + h(x^*) \\ &= y''(x^*, c^*) - \alpha''(x^*) + \alpha''(x^*) + f(x^*)\, \alpha'(x^*) + g(x^*)\, \alpha(x^*) + h(x^*) \\ &> y''(x^*, c^*) - \alpha''(x^*). \end{aligned}$$

This contradicts that $\alpha(x) \leqslant y(x, c^*)$ in $[-a, b]$. The proof of Lemma 2 is complete.

Now we can prove the main result in this paper.

THEOREM 3. *Under assumptions* (I) *and* (II), *for* $\varepsilon$ *in* $(0, \varepsilon_1]$ ($\varepsilon_1$ *some sufficiently small positive constant*) *the boundary value problem* (1), (2) *has a unique solution* $y(x, \varepsilon)$ *satisfying*

$$|y(x, \varepsilon) - Y_n(x, \varepsilon)| \leqslant C\varepsilon^{n+1}, \qquad -a \leqslant x \leqslant b, \tag{21}$$

*where* $C$ *is a positive constant independent of* $\varepsilon$ *and*

$$Y_n(x, \varepsilon) = \begin{cases} \bar{y}_0(x) + \sum\limits_{i=1}^{n} \left(\bar{y}_i(x) + \xi_i\left(\dfrac{x}{\varepsilon}\right)\right) \varepsilon^i, & 0 \leqslant x \leqslant b. \\ \bar{\bar{y}}_0(x) + \sum\limits_{i=1}^{n} \left(\bar{\bar{y}}_i(x) + \eta_i\left(\dfrac{x}{\varepsilon}\right)\right) \varepsilon^i, & -a \leqslant x \leqslant 0. \end{cases}$$

*Proof.* From the assumptions, there exist $\delta$, $\sigma$, $k$, $l$ and $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1]$, $g(x, \varepsilon) \leqslant -\sigma$ in $[-\delta, \delta]$, $|g(x, \varepsilon)| \leqslant l$ in $[-a, b]$, $f(x, \varepsilon) > k$ in $[\delta, b]$ and $f(x, \varepsilon) < -k$ in $[-a, -\delta]$ hold. Let

$$\bar{Y}_n(x, \varepsilon) = \begin{cases} Y_n(x, \varepsilon) + (2 - e^{-\sigma_1(x/\varepsilon)})\, \varepsilon^{n+1}, & 0 \leqslant x \leqslant b, \\ Y_n(x, \varepsilon) + e^{\sigma_2(x/\varepsilon)} \varepsilon^{n+1}, & -a \leqslant x \leqslant 0, \end{cases}$$

where $\sigma_1$ and $\sigma_2$ are two positive constants such that

$$\bar{y}_n'(0) + \sigma_1 = \bar{\bar{y}}_n'(0) + \sigma_2$$

holds. It follows that $\bar{Y}_n(x, \varepsilon)$ has continuous derivative at $x = 0$. In addition, by Theorem 2, for $x \in [0, b]$ we have

$$\begin{aligned}
&\varepsilon^2 \bar{Y}_n'' + f(x, \varepsilon)\, \bar{Y}_n' + g(x, \varepsilon)\, \bar{Y}_n + h(x, \varepsilon) \\
&\quad = \left[ \sum_{i=0}^{n} \bar{y}_i''(x)\, \varepsilon^{i+2} + f(x, \varepsilon) \sum_{i=0}^{n} \bar{y}_i'(x)\, \varepsilon^i \right. \\
&\qquad \left. + g(x, \varepsilon) \sum_{i=0}^{n} \bar{y}_i(x) + h(x, \varepsilon) + 2g(x, \varepsilon)\, \varepsilon^{n+1} \right] \\
&\qquad + \left[ \sum_{i=1}^{n+1} \xi_i''(\tau)\, \varepsilon^i + \frac{1}{\varepsilon} f(\tau\varepsilon, \varepsilon) \sum_{i=1}^{n+1} \xi_i'(\tau)\, \varepsilon^i + g(\tau\varepsilon, \varepsilon) \sum_{i=1}^{n+1} \xi_i(\tau)\, \varepsilon^i \right] \\
&\quad = O(\varepsilon^{n+1}),
\end{aligned}$$

where $\xi_{n+1}(\tau) = -e^{-\sigma_1 \tau}$. Similarly, for $x \in [-a, 0]$ we also have

$$\varepsilon^2 \bar{Y}_n'' + f(x, \varepsilon)\, \bar{Y}_n' + g(x, \varepsilon)\, \bar{Y}_n + h(x, \varepsilon) = O(\varepsilon^{n+1}).$$

Thus there exists a constant $M > 0$ such that for $x \in [-a, b]$,

$$|\varepsilon^2 \bar{Y}_n'' + f(x, \varepsilon)\, \bar{Y}_n' + g(x, \varepsilon)\, \bar{Y}_n + h(x, \varepsilon)| \leqslant M\varepsilon^{n+1}.$$

Define the functions

$$\begin{aligned}
\alpha(x, \varepsilon) &= \bar{Y}_n(x, \varepsilon) - w(x, \varepsilon), \qquad -a \leqslant x \leqslant b, \\
\beta(x, \varepsilon) &= \bar{Y}_n(x, \varepsilon) + w(x, \varepsilon), \qquad -a \leqslant x \leqslant b,
\end{aligned}$$

where

$$w(x, \varepsilon) = \begin{cases}
\varepsilon^{n+1} \Gamma l^{-1} (e^{\lambda_1 (x + a + \delta_0)} - 1), & -a \leqslant x \leqslant -\delta, \\
\varepsilon^{n+1} \Gamma l^{-1} (e^{\lambda_1 (a - \delta + \delta_0)} - 1), & -\delta \leqslant x \leqslant \delta, \\
\varepsilon^{n+1} \Gamma l^{-1} (e^{\lambda_1 (a - \delta + \delta_0)} - 1) & \\
\quad \times (e^{-\lambda_2 (b - \delta + \delta_0)} - 1)^{-1} (e^{-\lambda_2 (b - x + \delta_0)} - 1), & \delta \leqslant x \leqslant b.
\end{cases}$$

where $\delta_0$ is an arbitrarily given positive constant, $\Gamma$ is a positive constant to be chosen later, and $\lambda_1 = l/k + O(\varepsilon^2)$ is a positive root of $\varepsilon^2 \lambda^2 - k\lambda + l = 0$, $\lambda_2 = -l/k + O(\varepsilon^2)$ is a negative root of $\varepsilon^2 \lambda^2 + k\lambda + l = 0$.

It follows from the definition that $\alpha(x, \varepsilon) < \beta(x, \varepsilon)$, and for $\Gamma$ sufficiently large,

$$\alpha(-a, \varepsilon) < A(\varepsilon) < \beta(-a, \varepsilon), \qquad \alpha(b, \varepsilon) < B(\varepsilon) < \beta(b, \varepsilon).$$

Since $w'(x, \varepsilon) > 0$ in $[-a, -\delta)$ and $w'(x, \varepsilon) < 0$ in $(\delta, b]$, the functions $\alpha(x, \varepsilon)$ and $\beta(x, \varepsilon)$ satisfy

$$\alpha'(-\delta - 0, \varepsilon) \leqslant \alpha'(-\delta + 0, \varepsilon), \qquad \alpha'(\delta - 0, \varepsilon) \leqslant \alpha'(\delta + 0, \varepsilon),$$
$$\beta'(-\delta - 0, \varepsilon) \geqslant \beta'(-\delta + 0, \varepsilon), \qquad \beta'(\delta - 0, \varepsilon) \geqslant \beta'(\delta + 0, \varepsilon).$$

In addition, for $\Gamma$ sufficiently large and $\varepsilon$ in $(0, \varepsilon_1]$ we have

$$\begin{aligned}
&\varepsilon^2 \alpha''(x, \varepsilon) + f(x, \varepsilon)\,\alpha'(x, \varepsilon) + g(x, \varepsilon)\,\alpha(x, \varepsilon) + h(x, \varepsilon)\\
&\quad = \varepsilon^2 \bar{Y}_n''(x, \varepsilon) + f(x, \varepsilon)\,\bar{Y}_n'(x, \varepsilon) + g(x, \varepsilon)\,\bar{Y}_n(x, \varepsilon) + h(x, \varepsilon)\\
&\qquad - [\varepsilon^2 w''(x, \varepsilon) + f(x, \varepsilon)\,w'(x, \varepsilon) + g(x, \varepsilon)\,w(x, \varepsilon)]\\
&\quad \geqslant \begin{cases}
-M\varepsilon^{n+1} - [\varepsilon^2 w''(x, \varepsilon) - k w'(x, \varepsilon) + l w(x, \varepsilon)], \\
\qquad -a \leqslant x \leqslant -\delta \\
-M\varepsilon^{n+1} + \varepsilon^{n+1}\sigma \Gamma l^{-1}(e^{\lambda_1(a-\delta+\delta_0)} - 1), \\
\qquad -\delta \leqslant x \leqslant \delta, \\
-M\varepsilon^{n+1} - [\varepsilon^2 w''(x, \varepsilon) + k w'(x, \varepsilon) + l w(x, \varepsilon)], \\
\qquad \delta \leqslant x \leqslant b.
\end{cases}\\
&\quad = \begin{cases}
-M\varepsilon^{n+1} + \Gamma \varepsilon^{n+1}, \\
\qquad -a \leqslant x \leqslant -\delta, \\
-M\varepsilon^{n+1} + \Gamma \sigma l^{-1}(e^{\lambda_1(a-\delta+\delta_0)} - 1)\,\varepsilon^{n+1}, \\
\qquad -\delta \leqslant x \leqslant \delta, \\
-M\varepsilon^{n+1} + \Gamma(e^{\lambda_1(a-\delta+\delta_0)} - 1)(e^{-\lambda_2(b-\delta+\delta_0)} - 1)^{-1}\,\varepsilon^{n+1}, \\
\qquad \delta \leqslant x \leqslant b.
\end{cases}\\
&\quad > 0.
\end{aligned}$$

Similarly, for $\Gamma$ sufficiently large and $\varepsilon$ in $(0, \varepsilon_1]$, we have

$$\varepsilon^2 \beta''(x, \varepsilon) + f(x, \varepsilon)\,\beta'(x, \varepsilon) + g(x, \varepsilon)\,\beta(x, \varepsilon) + h(x, \varepsilon) < 0.$$

Now, $\alpha(x, \varepsilon)$ and $\beta(x, \varepsilon)$ satisfy all the assumptions of Lemma 2, thus the boundary value problem (1), (2) has a unique solution $y(x, \varepsilon)$ satisfying

$$\alpha(x, \varepsilon) < y(x, \varepsilon) < \beta(x, \varepsilon) \qquad \text{in} \quad [-a, b].$$

Hence it follows that there exists a constant $C > 0$ such that

$$|y(x, \varepsilon) - Y_n(x, \varepsilon)| \leqslant C\varepsilon^{n+1} \qquad \text{in} \quad [-a, b].$$

The proof is complete.

We note that there is yet a estimate about $|y'(x, \varepsilon) - Y_n'(x, \varepsilon)|$ in Theorem 2. To do this, we need the assumption that $f_1(x) \equiv 0$ which seems too strong. In fact, this is general in some extent. Consider the following boundary value problem

$$\varepsilon y'' + f(x, \varepsilon)\, y' + g(x, \varepsilon)\, y + h(x, \varepsilon) = 0,$$
$$y(-a) = A(\varepsilon), \qquad y(b) = B(\varepsilon),$$

where $f$, $g$, $h$, $A$, and $B$ satisfy assumption (I), and $f(0, 0) = 0$, that is, $x = 0$ is the turning point. We know that the solution has expansions with powers of $\varepsilon^{1/2}$. Setting $\mu = \varepsilon^{1/2}$ leads to

$$\mu^2 y'' + f(x, \mu^2)\, y' + g(x, \mu^2)\, y + h(x, \mu^2) = 0.$$

Let an expansion of $f(x, \mu^2)$ with powers of $\mu$ be

$$f(x, \mu^2) = f_0(x) + f_1(x)\mu + f_2(x)\, \mu^2 + \cdots,$$

then it is clear that $f_1(x) \equiv 0$. For this case, we have the following result.

THEOREM 4. *In addition to assumption* (I) *and* (II), *assume that* $f_1(x) \equiv 0$ *on* $[-a, b]$ *and there exists some odd number* $m > 1$ *such that* $f_0'(0) = \cdots = f_0^{(m-1)}(0) = 0$, $f_0^{(m)}(0) > 0$. *Then for* $\varepsilon$ *in* $(0, \varepsilon_1]$, *in addition to* (21),

$$|y'(x, \varepsilon) - Y_n'(x, \varepsilon)| \leqslant C_1 \varepsilon^n \exp[-\sigma_0 x^{m+1}/\varepsilon^2] + C_2 \varepsilon^{n-1+2/(m+1)}$$

*holds, where* $C_1$, $C_2$, *and* $\sigma_0$ *are positive constants independent of* $\varepsilon$.

*Proof.* We shall give the proof only for $x$ in $[0, b]$. The case for $[-a, 0]$ can be proved in a similar manner.

We know that $y(x, \varepsilon)$ satisfies

$$\varepsilon^2 y''(x, \varepsilon) + f(x, \varepsilon)\, y'(x, \varepsilon) + g(x, \varepsilon)\, y(x, \varepsilon) + h(x, \varepsilon) = 0. \tag{22}$$

In addition, let $Y_n(x, \varepsilon)$ satisfy the equation

$$\varepsilon^2 Y_n''(x, \varepsilon) + f(x, \varepsilon)\, Y_n'(x, \varepsilon) + g(x, \varepsilon)\, Y_n(x, \varepsilon) + h(x, \varepsilon) - \bar{h}(x, \varepsilon) = 0, \tag{23}$$

then $\bar{h}(x, \varepsilon) = O(\varepsilon^{n+1})$. This is because for $x \in [0, b]$,

$$\varepsilon^2 Y_n''(x, \varepsilon) + f(x, \varepsilon)\, Y_n'(x, \varepsilon) + g(x, \varepsilon)\, Y_n(x, \varepsilon) + h(x, \varepsilon) = O(\varepsilon^{n+1}).$$

Let $z(x, \varepsilon) = y(x, \varepsilon) - Y_n(x, \varepsilon)$. It follows from (22)–(23) that

$$\varepsilon^2 z''(x, \varepsilon) + f(x, \varepsilon)\, z'(x, \varepsilon) + g(x, \varepsilon)\, z(x, \varepsilon) + \bar{h}(x, \varepsilon) = 0.$$

By (21) and $\bar{h}(x, \varepsilon) = O(\varepsilon^{n+1})$,

$$\bar{\bar{h}}(x, \varepsilon) \triangleq g(x, \varepsilon)\, z(x, \varepsilon) + \bar{h}(x, \varepsilon) = O(\varepsilon^{n+1}). \tag{24}$$

In addition, we have

$$\begin{aligned} z'(x, \varepsilon) &= z'(0, \varepsilon) \exp\left[ -\frac{1}{\varepsilon^2} \int_0^x f(t, \varepsilon)\, dt \right] \\ &\quad + \frac{1}{\varepsilon^2} \int_0^x \bar{\bar{h}}(s, \varepsilon) \exp\left[ -\frac{1}{\varepsilon^2} \int_s^x f(t, \varepsilon)\, dt \right] ds. \end{aligned} \tag{25}$$

It follows from the assumptions that there exists a $\sigma_0$ sufficiently small such that

$$f_0(x) > (m+1)\, \sigma_0 x^m \qquad \text{in} \quad [0, b].$$

Thus for $0 \leqslant s \leqslant x \leqslant b$ and $0 < \varepsilon \leqslant \varepsilon_1$,

$$\int_s^x f(t, \varepsilon)\, dt \geqslant \int_s^x (f(t, \varepsilon) - f(t, 0))\, dt + \sigma_0 (x^{m+1} - s^{m+1}),$$

so by $f_1(x) \equiv 0$, there exists a positive constant $C_3$ such that

$$\exp\left[ -\frac{1}{\varepsilon^2} \int_s^x f(t, \varepsilon)\, dt \right] \leqslant C_3 \exp\left[ -\frac{\sigma_0}{\varepsilon^2} (x^{m+1} - s^{m+1}) \right]. \tag{26}$$

Since $x^{m+1} - s^{m+1} \geqslant (x-s)^{m+1}$ for $0 \leqslant s \leqslant x \leqslant b$, there exists a positive constant $C_4$ such that

$$\begin{aligned} \int_0^x \exp\left[ -\frac{1}{\varepsilon^2} \int_s^x f(t, \varepsilon)\, dt \right] ds &\leqslant C_3 \int_0^x \exp\left[ -\frac{\sigma_0}{\varepsilon^2} (x-s)^{m+1} \right] ds \\ &= C_3 \varepsilon^{2/(m+1)} \int_0^{x/\varepsilon^{2/m+1}} \exp[-\sigma_0 t^{m+1}]\, dt \\ &\leqslant C_4 \varepsilon^{2/(m+1)}. \end{aligned} \tag{27}$$

By Theorem 2,

$$z'(0) = O(\varepsilon^n). \tag{28}$$

Thus it follows from (24)–(28) that there exist positive constants $C_1$ and $C_2$ such that

$$z'(x, \varepsilon) \leqslant C_1 \varepsilon^n \exp[-\sigma_0 x^{m+1}/\varepsilon^2] + C_2 \varepsilon^{n-1+2/(m+1)}.$$

The proof is complete.

Now we consider an example of corner layer behavior. The boundary value problem

$$\varepsilon^2 y'' + xy' - y + 1 = 0, \tag{29}$$

$$y(-1) = A(\varepsilon), \qquad y(1) = B(\varepsilon), \tag{30}$$

has a unique solution $y(x, \varepsilon)$ by Theorem 3, and reduced solutions

$$\bar{y}_0(x) = (B_0 - 1)\, x + 1 \qquad \text{on} \quad [0, 1]$$

and

$$\bar{\bar{y}}_0(x) = (1 - A_0)\, x + 1 \qquad \text{on} \quad [-1, 0].$$

It is clear that $\bar{y}_0'(0) \neq \bar{\bar{y}}_0'(0)$ as $B_0 - 1 \neq 1 - A_0$. Thus the solution $y(x, \varepsilon)$ of the problem (29), (30) possesses a corner layer at the origin as $B_0 - 1 \neq 1 - A_0$.

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